Where Does Mathematics Fit?

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Introduction

Often when I mention to people that I'm a mathematician at a Christian liberal arts college, I can see their minds beginning to spin. Some thoughtful people begin to wonder what mathematics and Christianity have in common. Others ask how is "Christian math" different? After all, does not $1+1=2$ for non-Christians too? The quick answer is yes, but the longer answer puts this simple answer in perspective.

My students often share similar puzzlement. In fact, many expect no interaction between mathematics and Christianity. They can easily understand that Christianity can influence disciplines such as music, sociology, biology, *etc.*. But they are totally confused when asked what relevance Christianity has for mathematics.

Part of the puzzlement results from unfamiliarity with the nature of mathematics. Most people associate mathematics with numbers, algebra, calculus or even engineering. When they learned mathematics there was no need to invoke biblical texts or theological principles. Working with numbers was natural; algebra and calculus were just extensions of working with numbers. The intent of mathematics appears to most as nothing more than a tool for figuring things out. In the development of mathematics very little Christian thought made its way into the presentation and structure of mathematics. So many teachers are unaware of any connection between mathematics and religion.

In ancient times, there were mystical connections between religion and mathematics. For example, the Pythagoreans of ancient Greece held a mystical connection between numbers and music. In other eras, rather than religion influencing mathematics, mathematics influenced religion. For example, mathematics appears to be the language of natural laws: God created natural laws; therefore, God must be the supreme mathematician; hence, in order to understand nature you need to understand mathematics.

The puzzlement of many involves understanding what the discipline of mathematics studies. Very simplistically we can classify disciplines according to their object of study. For example: history, economics, sociology and political science study what motivates people and how they interact. Philosophy and psychology study how people think and reason. English, art, and music study creative works of mankind. The sciences, in some ways, can be thought of as the study of things. Chemistry studies molecules, biology studies living organisms and their interactions, and physics studies objects, some at the atomic scale others at the cosmic scale. People are puzzled when they are comfronted with the question "What do mathematicians study?"

Although people are puzzled by what mathematics is, they do know that in this technological age it has had a profound influence on their lives. Mathematics is powerful. So while trying to explain why Christianity is relevant to mathematics and while exploring the truth that exists in mathematics, we'll also briefly address why mathematics is so powerful? To begin we shall consider some fundamental aspects of Mathematics.

1 The Method of Mathematics

The method that sciences use to uncover truth in their disciplines involves both empirical evidence and reasoning (both deductive and inductive) from that evidence. Truth in mathematics rests on deductive reasoning, also called deductive logic.

We shall introduce deductive logic with a syllogism. A syllogism is a simple argument with three statements. The first two (called premises) supply information and the third is a conclusion based on the first two. Syllogisms date back to the the time of Aristotle. Consider the following argument:

p_1 All Canadians live in igloos.
p_2 Gary is Canadian.
$\Rightarrow q$ Gary lives in an igloo.

We will disregard the veracity of the statements and simply focus on whether the conclusion q holds when p_1 and p_2 are true. Intuitively we all know that the conclusion does hold, but how is this? It is based on our experiences of sorting and classifying objects. The statements imply three attributes to describe a person:

- Persons who are Canadian
- Persons who live in igloos
- Persons who are Gary.

Using these three attributes, we can classify all people. Person X is or is not Canadian. Person X does or does not live in an igloo. Person X is or is not Gary. When a person is sorted using these classifications, one of eight possibilities occurs (see the second to the last panel in figure 1). When the results of sorting with p_1 and p_2 are overlaid we

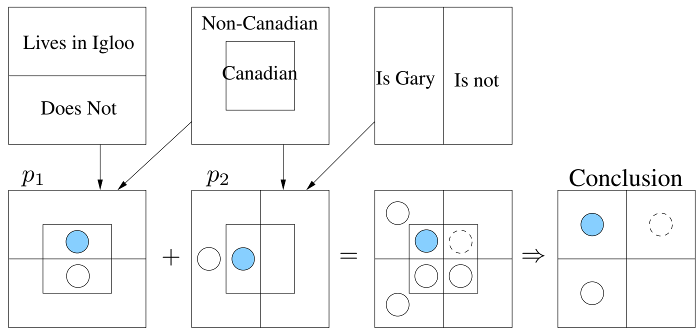

Figure 1: *The igloo argument in picture form. The top row represents sorting categories that hold in the lower row. Assuming the statements p_1 and p_2 are true then the shaded circles indicate categories that are known to be nonempty (assuming that at least one igloo, Canadian and Gary exist) and the open circles indicate known empty categories. The lower right two panels represent the statements p_1 and p_2 when combining attributes that the statements contain. Moving to the right, we see the results of combining the representations of p_1 and p_2. The dashed circles indicate uncertainty. From only statements p_1 and p_2 there is no way to know if there are any Canadians other than Gary, so from these statements we can not conclude what to do with the categories marked with a dashed circle. This method of proving syllogisms valid is credited to Lewis Carrol.*

see that the conclusion q is true.[1] Thus, we say that the argument presented is a true or valid argument.

A valid argument is one for which there is no possible way for the conclusion to be false when the premises are all assumed to be true. All truth in mathematics is arrived at by assuming some statements are true and use the methods of deductive logic to show that other statements must be true if the assumed ones are true.

There are many logical rules and methods to show arguments are valid; however, examining those would take us far from the purpose of this article. For our purposes it is enough to understand that mathematics at it's core is a collection of valid arguments that take the form: “If p_1, $p_2, \ldots$, p_n are all true then q must be true.”[2] That is, it is impossible for q to be false when p_1, $p_2, \ldots$, p_n are all true.

Some observations that can be made at this point are:

[1] Usually this is not considered strictly deductive reasoning. However, deductive reasoning is used in the sorting. For example, p_2 indicates that the inner box of Figure 1 has some thing in the left side, while p_1 indicates it is not in the lower half. So the conclusion drawn is that there is something in the upper left corner of the inner box. This is like saying: A or B is true and B is false, therefore A is true. In deductive logic this is called a disjunctive syllogism.

[2] This is even the case when working with algebra except the words are hidden. For example the statement “$3x + 7 = 10$ has solution $x = 1$” is really “If $x = 1$ then $3x + 7 = 10$.”

1. Deductive logic is based on experience and language of humans. The meaning of words like "*and*," "*or*," and "*not*" and phases like "*if* ... *then* ...," "*there exists* ...," *etc.* are at the core of deductive reasoning.

2. All mathematical statements should be either true or false. After all, it is not possible to be Canadian and not Canadian at the same time, or for $x = 2$ and $x \neq 2$ at the same time.

3. Valid arguments are not dependent on the veracity of the premises. In fact, in the argument above, p_1 and p_2 are false (if Gary refers to the author) and this had no bearing on the validity of the argument. The only concern is the validity or logical truth of the argument itself.

The question we now turn to arises out of the third observation. How can mathematics express truth if there is no concern about what is true?

2 Organization of Mathematics

Of course, mathematicians are concerned about what is true. In order to show that anything is true with deductive arguments you need two things:

1. A set of true statements (axioms) to argue from.
2. A method (deductive reasoning) to determine if arguments are valid.

We have touched on the second item in the previous section, and we noted that the methods used to show that arguments are valid are based on experience and language. If we have a set of statements that are known to be true based on experience then conclusions that can be drawn through valid arguments must be as true as our experiences since logic is also based on experience. If this were not the case then we would be forced to accept that our experiences do not always correspond to reality, at which point a large degree of certainty would be lost and we may as well be living in another reality completely apart from our experiences. The upshot of this is that if we believe that our experiences do correspond to reality then mathematics can contain truths about reality.

Figure 2: *Euclid, circa 300 BC.*

The process of organizing mathematical truth by using axioms and deductive reasoning started in ancient Greek times with geometry. Euclid (∼300 B.C.) summarized all known geometry of his time in a set of books called *The Elements.*[3] Euclid started

[3]The web site http://aleph0.clarku.edu/∼djoyce/java/elements/toc.html is a good source to learn more about *The Elements*.

with five geometric statements that he assumed were undeniable. They were close to these statements:

1. Two points determine a unique straight line.
2. A straight line may be infinitely extended.
3. A circle is determined by its center and a radius.
4. All right angles are the same.
5. There is only one line parallel to a given line passing through a given point.[4]

From statements like these (and some assumed common notions) he was able to supply arguments that established the truth of other known geometric statements. Two such statements that may be familiar from high school are shown in figures 3 and 4 below.

There is one other item that Euclid includes in *The Elements*; these are definitions of geometric objects. Many objects and notions were defined. Euclid even defined points and lines. Euclid defined points and lines in the following ways: "A point is that which has no part" and "A line is breadthless length." The idea was to define everything so that you could be absolutely sure about everything—is it or is it not an object.

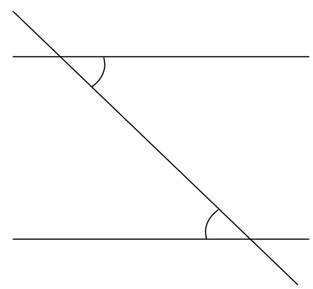

Figure 3: *If parallel line are crossed by a traversal then alternate interior angles are equal.*

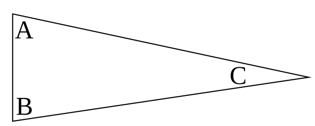

Figure 4: *The measures of the interior angles (A, B and C) of any triangle sum to 180°.*

Today modern mathematicians realize that it is not possible to define everything. For example one could ask Euclid "What is a "part?", "What is length?" or "What is breadth?" If everything requires a defintion, where would the definitions end? At some point it is required to accept that some things are not defineable; in the axiomatic system these things are called undefined terms. There are also common notions that we just know. For example: "The point is on the line" has the common notion of "on." Common notions are ultimately undefinable, yet we just know what they mean.

Undefined terms and common notions have whatever meaning our minds assign to them. This can be somewhat disconcerting, but as we shall see it is also very powerful and will help us understand the nature of mathematics.

Mathematics uses axiom systems to organize knowledge. Axiom systems have the three parts we have discussed.

[4]The statement that Euclid used was: "That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." The statement used above is known as "Playfairs axiom" has been shown to be equivalent to Euclid's fifth axiom and is easier to understand.

1. A set of common notions and undefined terms. All other objects must ultimately have their definitions in terms of undefined terms and common notions.

2. A set of axioms that are statements about the undefined terms that are assumed true. That is, statements that supply relationships between and establish the properties of the undefined terms.

3. A method of showing that an argument is valid (deductive logic).

Although Euclid did not recognize the need for undefined terms (only recognized in the last 300 years or so) "*The Elements*" formed the basis for nearly all mathematical thought for about 2000 years. You did not study mathematics unless you studied *The Elements.*

It seems that the truth represented in mathematical knowledge rests on undefined terms, common notions, statements assumed to be true and deductive logic, all of which are based on experience and human understanding.

Properties of axiomatic systems that will be important in our discussion are: *consistency*, *independence* and *completeness*. An axiomatic system is consistent if it is impossible, using deductive logic to arrive at a contradiction. A contradiction occurs when a statement can be deduced to be both true and false at the same time. If this is possible then deductive logic indicates that at least one of the axioms must be assumed false. Axiom systems that are not consistent are not worth consideration by mathematicians for this reason.

Ideally, every axiom in the set of axioms we assume true, should not follow as a consequence of the other axioms. If no axiom in the axiom set can be deduced from the other axioms, then the axioms in the set are said to be independent of one another. An axiom system is complete if all possible statements about objects in the system can be shown to be true or false.

It is a little mind-boggling to think about how to prove that no contradiction exists, that something is impossible to prove, or to think about all possible statements in a system. In the next section we will discuss these ideas. But first we introduce a simple axiom system that will be used to discuss the ideas of consistency and independence. The example is three point geometry. The axioms are:

A1: There exists exactly three points.
A2: Any two distinct points are contained in exactly one line.
A3: No line contains all points.
A4: Any two distinct lines contain at least one point in common.

Undefined terms are: *line* and *point*. Common notions are *contains*, *distinct*, and *exists*.

An example of a truth in this system is the statement that "*Two distinct lines contains **exactly** one point in common.*" This statement differs from axiom A4, since axiom A4 allows more than one point in common. Here is the proof: By axiom A4 there must be at least one common point contained in both lines, so there can not be distinct lines with no points in common. Suppose that there is more than one common point contained on both lines. Then the two lines must have at least two points in common. According to axiom A2 those two point determine exactly one line. So our "lines" must be a single line. This contradicts the assumption that we started with two distinct lines, so there cannot be more than one point in common.

Another statement that can be shown is "There are exactly three lines." This truth depends on all four axioms and its argument is more complicated. Statements or arguments that can be shown true from the axioms of the system are called theorems.

3 Models and Truth

We can now ask what truth beyond logical consistency exists in mathematical knowledge. This is particularly puzzling since it seems to depend on undefined terms and notions. Models of axiom systems help us understand the truth in mathematical knowledge.

A model of an axiomatic system is any realization of the system where all the axioms are true. A realization is an assignment of meaning or understanding to the undefined terms. For example, if you consider the axioms for three point geometry, then one model of the system is shown in the left panel of figure 5. A *point* means a dot on the page and a *lines* means the mark connecting dots. It is clear that all the axioms hold in the figure and as such all theorems that can be shown true must also hold.

An axiom system is shown to be consistent by illustrating a model for that axiom system. The model often is embedded in another axiomatic system that is assumed to be consistent. This of course begs the question: "Which axiomatic systems are known to be consistent?" I would propose that the certainty of consistency ultimately rests on one model: the physical creation. If a model of an axiomatic system is able to be found in the physical creation, then by virtue of the order that God endowed creation with, the model itself must be consistent.

The reader will carefully note that this assumes that it is impossible to have a state of affairs in creation that gives rise to a logical contradiction. This closely ties together mathematical knowledge and creation and it allows mathematics to uncover hidden parts of God's beautiful creation by exploring models through axiomatic systems. This accounts for the power and necessity of mathematics in the physical sciences.

Two cautions must be made at this point. The first is with the difficulties of finding a model in the physical world. It is generally not as easy as with three point geometry. Any axiom system that contains a concept that involves infinity (*e.g.* infinitely long straight lines, an infinite number of numbers, etc.) will be difficult, if not impossible, to model within the physical world. We will examine this in the context of geometry shortly.

The second caution is that it would be quite easy to say that the truth of mathematics rest on the physical world. But it is not clear what this would mean, since, for any axiom system there may be more than one model. This is possible by "redefining" the undefined terms. For example a second model for three point geometry may have *points* as persons and *lines* as committees (see center panel of figure 5). This ability to redefine the meaning of undefined terms gives mathematics its power to explain many diverse things that are seemingly unrelated. If the axioms hold in a particular model then the truth established in the axiom system extends to the model. The interplay between models and the consistency of axiomatic systems insure that this is the case. That is, the formalized reasoning of an axiom system that was based on experiences is repeatable within models where the assumptions of the axiom system hold. Thus, "experiences" with a model of an axiom system parallel the reasoning within the axiom system. Hence, the theorems of the axiom system hold in models of that system.

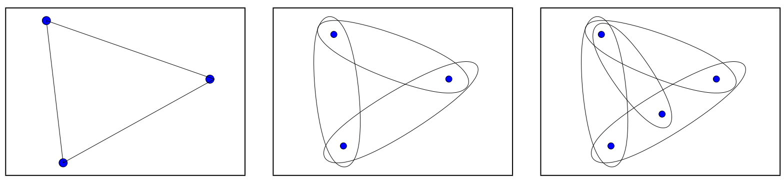

Figure 5: *Left: A model of three point geometry. Center: A second model of three point geometry. The dots represent persons (points) and the ovals represent committees (lines). Any theorem that is true in the left panel is also true in the center panel. Right: A model of four point geometry. This shows that the axiom A1 is independent of the other three axioms of three point geometry.*

Models are also used to show that axioms are independent. Suppose that in our list of three point geometry axioms we had included the statement A5: *Two distinct lines contains exactly one point.* This new list could easily serve as a set of consistent axioms for three point geometry; however, the new list would not be independent. A5 is dependent on A2 and A4 since it is provable from A2 and A4. That is, there is no way to have A5 false while A2 and A4 are true.

An axiom is shown independent of other axioms by displaying a model where that axiom is false and all the others are true.[5] For example, In three point geometry A1 is independent of the others, since there is a model with four points where A2, A3 and A4 are true (see right panel of figure 5). In this way the four axioms of our three point geometry can be shown to be independent.

The parallel axiom (5^{th} axiom) of Euclid's geometry is a statement that is more like a theorem when compared to the other axioms of Euclid. As a result, many people tried to "prove" it. If a proof of the parallel axiom was found then that make it a theorem so it could be removed from the axiom list.

[5] Strictly speaking these models must be in a consistent axiomatic system.

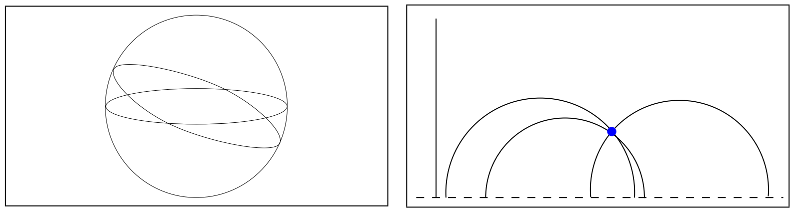

Figure 6: *Two lines are parallel if they do not intersect. Left: An example of elliptic geometry. In this example a "point" is any point on the surface of a sphere and a "line" is a great circle. There are no parallel line in this geometry since all great circles intersect. Right: An example of hyperbolic geometry. Here a "point" is any point in the half plane above the boundary line (dashed). A line is either a half line in the half plane perpendicular to the plane's boundary or a half circle in the plane whose center lies on the boundary. This is called the "Poincaré half plane". In this geometry there are an infinite number of parallel lines passing through a given point parallel to a given line. All the "lines" passing through the point are parallel to the "line" not passing through the point.*

Over time mathematicians[6] came to the realization that there are other possibilities. There could for instance be geometries where there are many or no lines passing though a given point parallel to a given line (see figure 6).

In these other geometries "strange" things are true. For example, in elliptic geometry (no parallel lines) the sum of the angles inside a triangle (an object bounded by three *lines*) is greater than $180°$ while in hyperbolic geometry (many parallel lines) the sum is less than $180°$ (see figure 6). However, in each geometry, if the area around a point is very highly magnified, all the geometries "look" similar. The consequence of this is that we do not know the geometry of the space that we living in. All we know is that locally it can be approximated with Euclid's geometry. The desire to have independent axioms to describe the geometry of the physical world has lead to the understanding that there maybe more than one axioms system that can describe things in the physical world.

A conclusion that we can make is that the truth in mathematics is an abstract truth which is not based on a physical model and, I have argued, is based on formalized human reasoning and language. It is not associated with the physical world until undefined terms are interpreted.[7] The truth of mathematics is often realized and confirmed with models which at best are only approximations of the physical world, since an exact correspondence between a model and the physical world is often impossible to know with certainty, as is the case of the goemetry of the physical world.

The third property of axiom systems that was mentioned in the previous section

[6]Saccheri (1667-1733), Lambert (1728 - 1777), Lobachewsky (1793-1856), and Bolyai (1802-1860) were instrumental in this realization.

[7]It should be noted that philosophers speak of alternate conceivable worlds, so this statement could be extended to any world. This of course gives rise to the question of where abstract truth exists, but we will leave this to the philosophers.

was completeness. An axiom system is complete if all statements about objects in the system can be shown to be true or false. In logic, valid statements are those that can be shown either true or false. As such, it is desirable to be able to determine the veracity of a statement.

In the late 1800s and early 1900s there was a movement in mathematics to establish a rigorous logical foundation of all mathematical truth. The goal was to take all mathematics and firmly establish it on axiomatic systems. Further, due to the successes that mathematics produced in physical sciences, mathematics was thought of as the final arbitrator of truth about the physical world. If something could be shown mathematically true then it was true. It came quite as a surprise in the 1930s that mathematics itself said that mathematics is limited in determining the truth of statements in axiomatic systems. Kurt Gödel in 1931 proved his famous incompleteness theorem that said it was impossible for any reasonably interesting consistent sytem to be complete. [8]. That is a consistent axiomatic system is incomplete. There are always be statements within an axiomatic system that can not be proven true or false.

Where mathematics had once been viewed as being able to determine truth in the physical world, we now know that even if we knew the "correct" axiom system of the physical world, there will still be questions that can not be answered within that axioms system. Thus, mathematical knowledge and knowledge about the physical world reveal through mathematics is ultimately limited since axiom systems can not be complete.

4 Created or Discovered?

Now that we understand better the nature of mathematics, we can address the question of where mathematics fits in God's creation. Often this is asked in the question "Is mathematics created or discovered?"

It appears that truths in mathematics are simply waiting for someone to ask: "Is statement X true?" This assumes that statement X is waiting to be found. Creating something implies an element of uniqueness that the creator bestows on an object. That is, if there is a different creator, then there is a different creation. Composition of music is a clear example where different composers create distinct sounds or themes in their music.

Whether or not a statement X is true is independent of the person or community who answers the question and only dependent on the axiom system. So there is an undeniable element of discovery when mathematicians make conjectures. There may be many ways to determine the veracity of a conjecture; each way is like a path to X through the logical landscape of known mathematics. Thus, proving a statement for the first time is like discovering a new path to a destination.

[8] http://www.time.com/time/time100/scientist/profile/godel.html gives an easy-to-understand summary of Gödel's incompleteness theorem.

Much of doing mathematics is discovering within an axiom system[9]; however, it is remiss to ignore the axioms. The combination of axioms that are chosen determines the mathematical landscape. While much of mathematics is exploring the landscape, the landscape is created by selection of axioms. It is possible for different mathematicians to manufacture or create different axioms, in a similar way composers create different musical themes. It is possible to create an entirely new axiom system that presents a new landscape to explore. The axioms used depend on who is putting the axioms together. Often the axioms are suggested by the physical world in the same way as sounds of nature may inspire a composer.

So what is the answer to the question "Is mathematics created or discovered?" I believe that it is both and both are done by humans. So where does mathematics fit in God's creation? I have argued that it is the creation of humans, since it is a human formalization of human reasoning based on human experiences. Generally mathematical knowledge is viewed to be trustworthy, since it's consistency is often tested by establishing correspondences with models in God's creation that are observed match the results in axiom systems. But as we have seen the correspondences are often not 100% certian. Thus, the knowledge about the physical world from mathematical is not 100% certain.

The difficulty that many have with calling mathematics "created" is the lack of tangibility in mathematics. We have seen that the truth in mathematics is abstract. Thus, it is not possible to touch, see, or hear it with fingers, eyes or ears. The problem many people have with considering mathematics "created" is not with the nature of mathematics but their limited definition of what "created" means. Mathematics at its foundational level is touched, seen, or heard with the mind since it is created with the mind.

A second difficulty that people have with calling mathematics "created by humans" is that they believe that this some how sets mathematics appart from God's creation. However, God created mankind in his own image and this endowed mankind with a creative ability. It could be called a "second order" creativity contained with the "first order" (God's creation). God provided the reasoning ability and experiences necessary for mankind to create mathematics. Mankind can not make physical things out of nothing, but mathematics is not a physical thing. It is abstract and not tangible until it is realized in a model.

There is an idea that existence of mathematical truth is independent of mankind. hence it can not be created by mankind. For example, some believe that principles of mathematics were in operation during creation before the existence of Mankind. Mathematics is, as we have seen, a product of human reasoning and experiences validated using models from creation. Thus, at best, I believe it can be said that during creation things were happening that may be describe mathematically. But in saying this, mathematics is treated as a human language, one that has been formulated and created to be consistent with order contained in models within creation. Thus, mathematics is very useful in describing creation. Since the models are within creation, it is natural that in

[9] Often the exact axiom system used is not explicitly stated and results are formally placed in an axiom system well after they are discoverd.

the ma themactics of diverse cultures common elements appear; making it look like mathematics is independent of mankind.

In conclusion, Mathematical truth is abstract human reasoning based on experiences that when realized in a model can provide a measure of truth about the model. This ability to gain knowledge of a model demonstrates the power that is present in mathematics and is a direct result of the order God created in all of creation.